Assignment 8

1. Suppose f is a real-valued function on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leqslant (x - y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is a constant function.

2. Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then show that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

3. Suppose $a \in \mathbb{R}$, f is a twice differentiable real-valued function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a, ∞). Prove that

$$M_1^2 \leqslant 4M_0M_2.$$

4. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is real number A such that $|f'(x)| \leqslant A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

5. Recall that for a uniformly continuous function f on $\mathbb{R}$, we have

$$|f(x)| \leqslant A + B|x - x_0|$$

for some constants A, B depending on x_0. Find a uniformly continuous function g on $\mathbb{R}$ such that for all $x_0 \in \mathbb{R}$

$$|g(x)| \leqslant |g(x_0)| + B|x - x_0|$$

does not hold for any B.

6[1]. Suppose f is a real analytic function on $\mathbb{R}$ such that for each $x \in \mathbb{R}$, we have $n \in \mathbb{N}$ such that $f^{(n)}(x) = 0$. Show that f is a polynomial. *Hint*: Use Baire's Category Theorem.

[1] Need not to be submitted